Appendix to "Balloon-Supported Vertical Arrays for 160 Meters," by P.M. Livingston, W3CRI; Dave Kunkee, K0DI, and Elizabeth Kunkee, KS4IS, in the February 2003 issue of *CQ Amateur Radio* magazine. This appendix is part of the article and is Copyright © 2002-2003 CQ Communications Inc.

APPENDIX A.
MATHEMATICAL DETAILS

Figure A-1 is an equivalent circuit diagram of the interacting elements of our phased array.

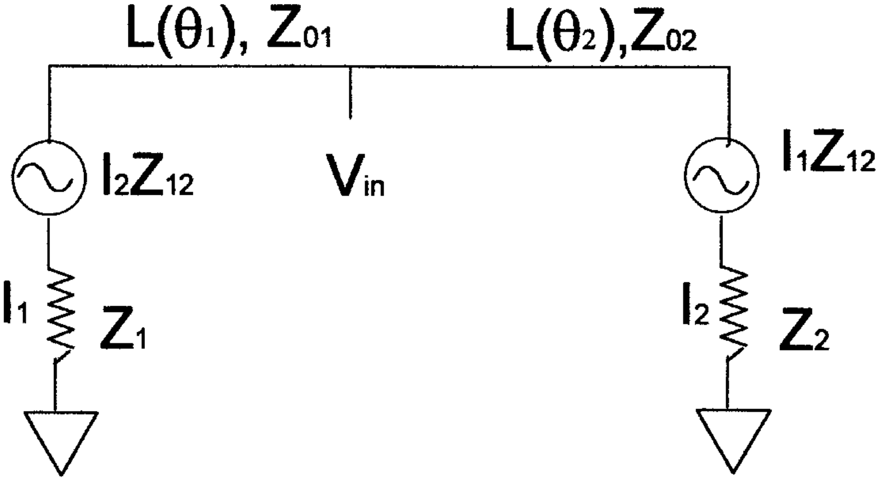

Figure A-1. A lumped circuit network that represents a mutual-impedance coupled phased array. The two phasing line lengths are denoted by $L(\theta_1)$ and $L(\theta_2)$ with impedances Z_{01} and Z_{02}.

The source, "V_{in}" is the common feed from the transmitter. Each antenna has a basic self-impedance Z_1, Z_2 consisting of wire, radiation resistance and ground losses. However the voltage across each antenna is the sum of the IZ drop as shown plus that coupled from the other antenna denoted by $I_{1,2}Z_{12}$. This latter contribution to the total voltage drop is substantial and will lead to poor array performance if ignored. Hence the design problem is to adjust the electrical lengths of the phased array feed lines to (a) match the antenna currents and (b) provide a 90-degree phase shift between the two currents.

$$\frac{I_2}{I_1} = \frac{Z_1 \cos\theta_1 - Z_{12,} \cos\theta_2 + jZ_{01} \sin\theta_1}{Z_2 \cos\theta_2 - Z_{12} \cos\theta_1 + jZ_{02} \sin\theta_2} \tag{A-1}$$

This complex equation, valid for loss-free transmission lines, can be solved for the two angles (the phasing line lengths are expressed in degrees), by first requiring the two currents to be equal and 90 degrees out of phase with $Z_{01}= Z_{02}$ being the phasing line impedance. A good estimate for the antenna self impedances is 55 ohms at resonance, which is consistent with our measurements. It turns out that there is no solution for 50 ohm phasing line impedance, but there is if $Z_{01} = Z_{02} = 75$ ohms. Next we must insert an estimate for the mutual impedance. As a first guess, we used data taken from figure 19, page 8-15 of the *ARRL Antenna Book* , 17th Edition for the mutual resistance and reactance of a quarter-wave phased array. Values chosen for the computation were 20 and –15 ohms, respectively.

The algebra for solving equation (A-1) is tedious and consists of these steps: (a) Rationalize the right hand side by multiplying numerator and denominator by the complex conjugate of the denominator; (b) set the ratio of the currents to $j = \sqrt{-1}$; and (c) separate the equation into its real and imaginary parts. I put together a MATHCAD program that solves for the electrical phasing line lengths that is listed in Appendix B with a MATHCAD root-finding algorithm listed here in Appendix A. Even if you don't have MATHCAD, the formulae can be programmed in other languages. The computed values of 69.41 and 154.65 degrees are very close to values given in Lewallen's article (see main text for citations/references) for an eight radial ground plane. Next I inserted these values into EZNEC for the feedline lengths and adjusted the lengths to place the overall antenna resonance in the lower part of the 160 meter band where most of the activity will be. The adjusted values are 53.41 and 155.36 degrees respectively that also reflects the change in mutual impedance arising from the slanted antennae. It turns out that differences of several degrees in electrical length did not make a large difference in antenna performance.

While the algebra for the reduction of equation A-1 is not particularly abstruse, it does not appear in any journal that I am aware of. Accordingly, mathematically inclined hams may find the discussion below useful and informative.

Equation (A-1) is easily reorganized into the following ratio of complex numbers:

$$\begin{aligned}
&Me^{j\varphi} = \frac{A - jB}{C - jD} = \frac{AC + BD + j(AD - BC)}{C^2 + D^2} \\
&A = R_a \cos\theta_1 - R_{12} \cos\theta_2; \\
&B = \mathrm{X}_{12} \cos\theta_2 - R_f \sin\theta_1; \\
&C = R_b \cos\theta_2 - R_{12} \cos\theta_1; \\
&D = \mathrm{X}_{12} \cos\theta_1 - R_f \sin\theta_2
\end{aligned} \tag{A-2}$$

Note the second equality expresses a rationalization of the complex ratio, formed by multiplying both the numerator and denominator by the complex conjugate of the denominator. (Replacing j everywhere by –j forms the complex conjugate of the expression.)

The current ratio of the two antennae (base impedances R_a, R_b) is M and the phase difference is φ. Our phased array requires M = 1 and φ = π/2. Accordingly, eq. A-2 is separable into two components by equating real and imaginary parts on both sides of the equation. These are:

$$\begin{aligned} AC + BD &= 0; \\ AD - BC &= C^2 + D^2 \end{aligned} \qquad \text{(A-3)}$$

From these two expressions, it is easy to prove that A = D and –B = C, given that the sum of squares of C and D are nonzero. From the second set of equations shown in (A-2) and the equalities just given, we can derive an expression that involves only $\cos(\theta_1)$ (or $\cos(\theta_2)$). To simplify the mathematics, let $y = \cos(\theta_1)$, $x = \cos(\theta_2)$, and

$$a = \frac{R_{12}}{R_b + \mathrm{X}_{12}}; \quad b = \frac{R_f}{R_b + \mathrm{X}_{12}}; \quad c = \frac{R_{12}}{R_a - \mathrm{X}_{12}}; \quad d = \frac{R_f}{R_a - \mathrm{X}_{12}}. \qquad \text{(A-4)}$$

Following the substitutions just given, we derive two linear equations in x and y that are easily reduced to a single expression. The two equations are:

$$\begin{aligned} y &= cx - d(1 - x^2)^{1/2} \\ x &= ay + b(1 - y^2)^{1/2} \end{aligned} \qquad \text{(A-5)}$$

Substitute the first expression for y in the second equation, and, recognizing that ad = bc, we obtain the eighth-order equation shown in the MATHCAD listing, below.

$$f(x) = x(1 - ac) + bc(1 - x^2)^{1/2} - b\left[1 - (cx - d(1 - x^2)^{1/2})^2\right]^{1/2} = 0\,. \qquad \text{(A-6)}$$

We are interested only in the real roots of this equation that lie between 0 and 1. There should be two. Use a root-finding routine to determine x, then substitute its value in either one of the equations shown in A-5 to obtain y. Clearly, the values of the mutual impedance of the array must be known as well as the antenna base impedances at resonance and of course, the phasing line impedance.

By a process of trial and error, you can use EZNEC to determine the phasing line electrical lengths such that your antenna array has a maximum forward to back ratio at a particular frequency. Now you may wish to determine the mutual impedance that gave rise to the particular phasing line lengths. From equations A-4 and A-5 one can derive expressions for the real and imaginary parts of the antenna array mutual impedance:

$$\begin{aligned} R_{12} &= \frac{R_b y}{\frac{y^2}{x} + x} - \frac{R_f y(1 - y^2)^{1/2}}{y^2 + x^2} + \frac{R_a y}{\frac{y^2}{x} + x} + \frac{R_f (1 - x^2)^{1/2}}{\frac{y^2}{x} + x} \\ \mathrm{X}_{12} &= \frac{R_{12} y}{x} - R_b + \frac{R_f (1 - y^2)^{1/2}}{x} \end{aligned} \qquad .\text{(A-7)}$$

For the two-element phased array described in the article, given the optimum phasing line lengths of 155.36 and 53.41degrees, respectively, using these equations for a nominal 50 ohm antenna base impedance and a 75 ohm phasing line, I determined that R_{12} = 8.6 ohms and X_{12}= -10.63 ohms. For parallel quarter-wave phased antennae, a plot in Krause's book[1] shows that the real part of the mutual impedance is about 20 ohms and the imaginary part is –15 ohms.   Therefore tilting the antennae towards one another causes a substantial change in the mutual impedance as might be expected.

Note: 1 - J. D. Krause, *Antennas* ,McGraw-Hill, (1988), page 426, Figure 10-12.

APPENDIX B
MATHCAD LISTING FOR PHASING LINE CALCULATION

This program computes feedline lengths for a phasing line to feed two quarter-wave verticals 90 degrees out of phase.

$R_a := 55$ resonant impedance at the base of one vertical (the other disconnected)

$R_{12} := 20 \quad X_{12} := -15$ Real and imaginary parts of the mutual impedance

$R_f := 75$ Feedline impedance

$$a := \frac{R_{12}}{\left(R_a + X_{12}\right)} \qquad b := \frac{R_f}{\left(R_a + X_{12}\right)} \qquad c := \frac{R_{12}}{\left(R_a - X_{12}\right)} \qquad d := \frac{R_f}{\left(R_a - X_{12}\right)}$$

$$F(x) := (1 - c \cdot a) \cdot x + b \cdot c \cdot \left(1 - x^2\right)^{\frac{1}{2}} - b \cdot \left[1 - \left[c \cdot x - d \cdot \left(1 - x^2\right)^{\frac{1}{2}}\right]^2\right]^{\frac{1}{2}}$$

$x := 0.1$ initial guess for the root finder.

$x_{root} := \mathrm{root}(F(x), x)$

$x_{root} = 0.349$

$\theta_2 := \mathrm{acos}\left(\mathrm{Re}\left(x_{root}\right)\right) \qquad \frac{\theta_2}{2 \cdot \pi} \cdot 360 = 69.546$ Phasing line 2 length

$$x_1 := c \cdot x_{root} - d \cdot \left(1 - x_{root}^{\ 2}\right)^{\frac{1}{2}}$$

$\theta_1 := \mathrm{acos}\left(x_1\right) \qquad \frac{\theta_1}{2 \cdot \pi} \cdot 360 = 154.694$ Phasing line 1 length

$$A := R_a \cdot \cos\left(\theta_1\right) - R_{12} \cdot \cos\left(\theta_2\right)$$

$$B := X_{12} \cdot \cos\left(\theta_2\right) - R_f \cdot \sin\left(\theta_1\right)$$

$$C := R_a \cdot \cos\left(\theta_2\right) - R_{12} \cdot \cos\left(\theta_1\right)$$

$$D := X_{12} \cdot \cos\left(\theta_1\right) - R_f \cdot \sin\left(\theta_2\right)$$

CHECK

$A = -56.711 \qquad B = -37.301 \qquad C = 37.3 \qquad D = -56.711$

Impedance at the feedpoint of the phasing line

$$Z_2 := \left(iR_a + R_{12} + i \cdot X_{12}\right) \cdot \cos\left(\theta_2\right) - R_f \cdot \sin\left(\theta_2\right)$$

$Z_2 = -63.283 + 13.978i$

$$\rho := \sqrt{\frac{\left(-\mathrm{Re}\left(Z_2\right) - 50\right)^2 + \mathrm{Im}\left(Z_2\right)^2}{\left(-\mathrm{Re}\left(Z_2\right) + 50\right)^2 + \mathrm{Im}\left(Z_2\right)^2}} \qquad \rho = 0.169$$

$$SWR := \frac{(1 + \rho)}{1 - \rho} \qquad SWR = 1.407$$